Acta Informatica 37, 1–19 (2000)

Semantics analysis through elementary meanings

Theoretical foundation for generalized thesaurus construction

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Received: 5 May 1996 / 21 February 2000

Abstract. Thesaurus is a collection of words classified according to some relatedness measures among them. In this paper, we lay the theoretical foundations of thesaurus construction through elementary meanings of words. The concept of elementary meanings has been advocated and utilized in compiling Webster's Collegiate Thesaurus. If each word is supplied with elementary meanings so that all its meanings are covered by them in a standard fashion, we can define various similarity measures for a given set of words. Here we take an axiomatic way to analyze semantic structure of word groups. Assuming an abstract semantic world, we deduce closed sets as generalized synonym sets. That is, we show that under certain natural axioms, we only need to consider closed sets as far as the semantics are concerned. We also show that the set of generalized synonyms described as a certain pair of closed sets has a lattice structure. In order to have a flexible thesaurus, we also analyze structure changes corresponding to three basic environmental changes: A new word-meaning relation is added, a new word or a new meaning is included with its word-meaning relations. Actually we give algorithms to have updated lattice structure from previous one for the three operations.

This work has been supported by Special Coordination Funds for Promoting Science and Technology (Science and Technology Agency, Japan) as a part of research project "Self Organizing Information-Base Systems for Creative Research and Development (1991–1995)."

1 Introduction

Thesaurus is a collection of words classified according to some relatedness measures among them. The relations include synonyms, antonyms, broader terms, narrower terms, and so forth. These relatedness relations are often given subjectively at certain fixed level and are difficult to treat quantitatively. This is particularly so when thesauri for general terms are concerned. We want to establish a systematic method to construct objective, flexible, and versatile thesauri as automatically as possible. The basic difficulty stems from the fact that we have to handle various meanings of each word. When we consider general terms, meanings of a word are supposed to be described in an ordinary dictionary. But how can we compare these defining meanings preferably in an automatic fashion? One drawback of ordinary definitions of words in a dictionary (for our purpose at least) is that they are mostly not written for exact comparison in mind. Here consideration for exact comparison means, for example, that the same expression should be used to define the same meaning if it is applicable.

Historically speaking, the first modern English thesaurus was compiled using top-down method by Roget [9] who first gave six classes like Abstract relations, Space, Matter and so forth. Then, he selected nearly one thousand head words under which various words and phrases were allocated. Note that his classification has been done by his own understanding of the meanings depending on his personal feelings about the words and phrases. The situation seems similar for a more recent compilation of a Japanese thesaurus [7]: The editors first set up ten categories and each category is divided into ten subcategories and finally one thousand head words are selected in *ad hoc* fashion. This traditional method relies heavily on editors' capabilities and intuition, and it takes much time and energy to complete and maintain resulting thesauri.

A component analysis method [4], on the other hand, defines a set of microfeatures to classify a given set of words or objects. This is based on the standard idea in philosophy to define concepts. The idea has been utilized in knowledge representation systems for some AI researches [6, 8]. That is, each concept has its intent and extent where intent corresponds to microfeatures and extent means a collection of objects or words belonging to that concept. A lattice theoretic investigation along this line has also been carried out by Ganter and Wille [3]. In this method, a context under consideration is given *a priori* and analysis is done by selecting (subjectively) some features to identify the objects belonging to the context. Through this microfeature description, words can be classified according to the different feature values they assume. We can't be sure, however, if certain pair of words have some common meaning even when they so far fell into the same categories.

In compiling a thesaurus, a bottom-up method is also feasible. A typical way to do is to give explicit words relations such as synonyms, broader terms, antonyms, and so on. This will work as much the same way as the traditional top-down method but also share the similar drawbacks. An automatic thesaurus construction technique for information retrieval systems uses statistical methods to extract such words relations [10]. In real elaborate thesaurus construction such as the Concept Dictionary by Japan Electronic Dictionary Research Institute [2], both bottom-up and top-down methods have been employed.

We here consider another bottom-up method which uses *elementary meanings* of words. The idea of elementary meanings has been advocated and utilized in compiling Webster's Collegiate Thesaurus [5]. It is a standardized way of giving meanings under which we can decide certain set of words become synonymous. These elementary meanings in the thesaurus should be considered as examples and should not be regarded as a unique way nor a standard way of giving them. One of the important achievements of Webster's Collegiate Thesaurus is that it showed that elementary meanings can be definable at least to an extent that reasonable size thesaurus is compiled using them. In this paper, we don't try giving appropriate elementary meanings to a particular set of words, which will be done elsewhere. Rather, we analyze semantic structure of word sets under the assumption that some suitable elementary meanings are given. To make our idea as clear as possible, we restrict ourselves to the case where only broader/narrower relations (and hence equality relation) are taken into account.

In Sect. 2, we give necessary definitions for our framework and introduce closed sets. To clarify the relations between words and elementary meanings, we consider abstract semantics world in Sect. 3. We set up several natural axiomatic relations and reveal some basic properties of our universe, which is the relations between word set and elementary meaning set. In so doing, we introduce *independent systems*. In Sect. 4, a pair of closed set is extracted as a *generalized synonym*. Then for the set of generalized synonyms, we have lattice structure called *semantic similarity lattice*, which is actually a *generalized thesaurus*. Here a generalized thesaurus is a thesaurus where hierarchically structured word sets (generalized synonyms) are arranged graphically. In these generalized synonyms, words are grouped at various relatedness levels according to the corresponding elementary meaning sets. In Sect. 5, we define three basic operations which correspond to the elemental changes of our universe. That is, add word, add meaning, and add relation operations. We give an algorithm to update the generalized thesaurus under each of the three operations. Although we analyze the properties of the words-elementary meanings relation theoretically, we also use elementary meanings of a few words from Webster's Collegiate Thesaurus for explanatory purpose. Detailed validity proofs of the update algorithms are shown in the Appendix.

2 Words and their meanings – A binary relation

We begin by presenting a formal definition of our framework and its mathematical consequences. First, we assume a binary relation A on the set of words W and the set of elementary meanings M: $A \subseteq W \times M$. Intuitively, for $w \in W$ and $m \in M$, $(w, m) \in A$ is intended to mean that the word w has a meaning m. We assume that for any $w \in W$ there is some $m \in M$ such that $(w, m) \in A$, and for any $m \in M$ there is some $w \in W$ such that $(w, m) \in A$. We denote this system as (W, M; A) and call it as the *universe* or our universe of discourse.

Define two functions μ: $W \to 2^M$ and ω: $M \to 2^W$ by the relation A as follows.

$$\mu(w) = \{m \in M | (w, m) \in A\};$$
$$\omega(m) = \{w \in W | (w, m) \in A\}.$$

We extend these functions for the domains 2^W and 2^M, respectively, as follows.

$$\mu^* : 2^W \to 2^M \quad \text{where} \quad \mu^*(U) = \bigcap_{w \in U} \mu(w);$$
$$\omega^* : 2^M \to 2^W \quad \text{where} \quad \omega^*(K) = \bigcap_{m \in K} \omega(m).$$

For completeness' sake, we assume that $\mu^*(\emptyset) = M$ and $\omega^*(\emptyset) = W$. By the definitions, it is easy to see the followings [1, 3].

Lemma 1. *For any $U_1, U_2 \subseteq W$, $U_1 \subseteq U_2$ implies $\mu^*(U_2) \subseteq \mu^*(U_1)$, and for any K_1 , $K_2 \subseteq M$, $K_1 \subseteq K_2$ implies $\omega^*(K_2) \subseteq \omega^*(K_1)$.*

Lemma 2. *For any $U_j \subseteq W$ and $K_j \subseteq M (j \in J)$, we have*

$$\mu^* \left(\bigcup_{j \in J} U_j \right) = \bigcap_{j \in J} \mu^*(U_j) \quad \textit{and} \quad \omega^* \left(\bigcup_{j \in J} K_j \right) = \bigcap_{j \in J} \omega^*(K_j).$$

2.1 Closed set

Consider an ordered set $(L, \geq)$ and a mapping φ from L into itself. If the mapping satisfies the following three conditions, then it is called a *closure operator* [1, 3].

(1) $\varphi(x) \geq x$
(2) $\varphi(\varphi(x)) = \varphi(x)$
(3) $x \geq y \Rightarrow \varphi(x) \geq \varphi(y)$

An element x is called *closed* if $\varphi(x) = x$. When L is the power set of a set X, then a closed element of L is also said to be a *closed set* of X.

It is easy to see that $\omega^*\mu^*$ and $\mu^*\omega^*$ are closure operators on 2^W and 2^M, respectively. In fact, the binary relation A with mappings μ^* and ω^* is known as a Galois pair [1, 3]. Thus we have the following basic results for the closed sets with respect to these operators.

Lemma 3(a). *For any $U \subseteq W$, the following four assertions are mutually equivalent.*

(1) *U is a closed set.*
(2) $\omega^*\mu^*(U) = U$.
(3) *$U = \omega^*(K)$ for some closed $K \subseteq M$.*
(4) *$U = \omega^*(K)$ for some $K \subseteq M$.*

Lemma 3(b). *For any $K \subseteq M$, the following four assertions are mutually equivalent.*

(1) *K is a closed set.*
(2) $\mu^*\omega^*(K) = K$.
(3) *$K = \mu^*(U)$ for some closed $U \subseteq W$.*
(4) *$K = \mu^*(U)$ for some $U \subseteq W$.*

A closed set $\omega^*(K)$ is to correspond to the set of words having all the meanings of K. Then the words in $\omega^*(K)$ can be regarded as synonymous as far as the meanings K are concerned. In later sections, the pair ($\omega^*(K)$, K) is called as a generalized synonym if K is closed. In fact, Webster's Collegiate Thesaurus [5] is a thesaurus which is basically listing words of $\omega^*(m)$ where $\{m\}$ is a singleton elementary meaning. The pair ($\omega^*(m)$, $\{m\}$) may be called as an elementary synonym, which is not necessarily a generalized synonym. These concepts will be made clear in the following sections.

3 A simple model of semantics

Let R denote an abstract world of semantics in which various relations are given *a priori*. Typical such relations include identical, broader/narrower, related, antonym relations and so forth. To make our framework as simple as possible for explanatory purpose, we consider here only broader/narrower relations as a basic relation. That is, we consider a partially ordered set (R $\sqsubseteq$) where for any elements $\alpha, \beta \in R$, α is broader than β (or equivalently β is narrower than α) if $\beta \sqsubseteq \alpha$ or $\alpha \sqsupseteq \beta$. Note that $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$ imply $\alpha = \beta$. Intuitively speaking, α is broader than β if α represents all and every semantics represented by β.

3.1 Fundamental relations

For a given universe $(W, M; A)$, we here think of two mappings $\Pi : 2^W \to R$ and Σ: $2^M \to R$ which relate a set of words and a set of elementary meanings to the world of semantics, respectively. We call them as *semantics specification mappings* for W and M, respectively. Given a subset U of W, we associate the semantics ΠU to it. Our intention is that this semantics is the one shared by all the words in U. For a subset K of M, we associate the semantics ΣK which has all the meanings of K. The above mentioned interpretations of the two mappings are given here solely for explanatory purposes. The properties of Π and Σ are to be derived from the following axiomatic relations.

Semantics Axiom. A–1. $\Pi U = \Sigma \mu^*(U)$ *for any* $U \subseteq W$.

Consider a word w, then the Semantics Axiom implies $\Pi w = \Sigma \mu(w)$. This means that the semantics of w should be exactly covered by the elementary meanings $\mu(w)$. This is one of the basic premises when we treat elementary meanings of words.

Monotony Axiom. A–2. $K \subseteq L$ *implies* $\Sigma K \sqsubseteq \Sigma L$ *for any* $K, L \subseteq M$.

The Monotony Axiom simply states that the more elementary meanings there are, the broader semantics they carry. This poses certain restriction on how the set of elementary meanings should be selected.

We can deduce some immediate consequences from the above axioms as follows.

For any $K \subseteq M$, $\Pi \omega^*(K) = \Sigma \mu^*(\omega^*(K))$ by A–1 and $\Sigma K \sqsubseteq \Sigma \mu^* \omega^*(K)$ by A–2. This means

Lemma 4. *$\Sigma K \sqsubseteq \Pi \omega^*(K)$ for any $K \subseteq M$, and $\Sigma K = \Pi \omega^*(K)$ for any closed $K \subseteq M$.*

For a pair of subsets of W, we can see that $U \subseteq V \Rightarrow \mu^*(V) \subseteq \mu^*(U) \Rightarrow \Sigma \mu^*(V) \sqsubseteq \Sigma \mu^*(U) \Rightarrow \Pi V \sqsubseteq \Pi U$. That is, we have

Lemma 5. *$U \subseteq V$ implies $\Pi V \sqsubseteq \Pi U$ for any $U, V \subseteq W$.*

For any two words w_1, $w_2 \in W$, w_1 is said to be a *broader term* than w_2 (written as $w_1 \succcurlyeq w_2$) if $\Pi w_1 \sqsupseteq \Pi w_2$.

We have the following lemmas. First we show a natural property that w_1 is a broader term than w_2 if and only if w_1 has more elementary meanings than w_2.

Lemma 6. *For any two words $w_1, w_2 \in W$, $\mu(w_1) \supseteq \mu(w_2)$ if and only if $w_1 \in \omega^*\mu^*(w_2)$ and these relations imply $w_1 \succcurlyeq w_2$.*

Proof. $\mu^*(w_1) \supseteq \mu^*(w_2) \Rightarrow \Sigma\mu^*(w_1) \sqsupseteq \Sigma\mu^*(w_2) \Rightarrow \Pi w_1 \sqsupseteq \Pi w_2$. On the other hand, $\mu^*(w_1) \supseteq \mu^*(w_2) \Rightarrow \omega^*\mu^*(w_1) \subseteq \omega^*\mu^*(w_2)$ which means that $w_1 \in \omega^*\mu^*(w_2)$. Conversely, $w_1 \in \omega^*\mu^*(w_2)$ implies $\mu^*(w_1) \supseteq \mu^*(w_2)$. □

Lemma 7. *$(w, m) \in A$ implies $\Sigma\mu^*\omega^*(m) \sqsubseteq \Pi w$ for any $w \in W$ and $m \in M$. Then $(w, m) \in A$ implies $\Sigma m \sqsubseteq \Pi w$ a fortiori.*

Proof. Let $(w, m) \in A$ for $w \in W$ and $m \in M$. Then $(w, m) \in A \Rightarrow \{w\} \subseteq \omega^*(m) \Rightarrow \mu^*(w) \supseteq \mu^*\omega^*(m) \Rightarrow \Sigma\mu^*(w) \sqsupseteq \Sigma\mu^*\omega^*(m) \Rightarrow \Pi w \sqsupseteq \Sigma\mu^*\omega^*(m)$. □

From the Semantics Axiom and Lemma 4, we have

Lemma 8. $\Pi U = \Sigma\mu^*(U) = \Pi\omega^*\mu^*(U)$ *for any* $U \subseteq W$.

As $\omega^*\mu^*(U)$ is the closure of U, this means we only need to consider closed subsets of W in the universe ($W, M; A$) as far as the semantics of R are concerned. In other words, closed sets are not only induced by Galois pair but also deduced naturally from axiomatic semantics analysis.

3.2 Independent systems

Now, we introduce a property which is the converse of A–2 for closed sets, and call it as independence. That is, we assume, in the sequel, the following independence axiom.

Independence Axiom. A–3. $\Sigma K \sqsubseteq \Sigma L$ implies $K \subseteq L$ for any closed $K, L \subseteq M$.

Independence then means that $\Sigma K = \Sigma L$ implies $K = L$ when K and L are both closed. In other words, independence requires that distinct closed sets of elementary meanings have different semantics. This also implies that the μ function is unique under given W, M and R. That is, if there were μ and μ' such that $\mu(w) \neq \mu'(w)$ for some $w \in W$, then the independence axiom deduces a contradiction. Thus the relation between W and M has less flexibility such that ambiguities like multiple meanings may not exist. Immediate consequences of assuming independence are as follows.

Lemma 9.

(a) *$\mu^*(U) \supseteq \mu^*(V)$ is equivalent to $\Pi U \sqsupseteq \Pi V$ for $U, V \subseteq W$.*
$\mu(w_1) \supseteq \mu(w_2)$ is equivalent to $w_1 \succcurlyeq w_2$ in particular.
(b) *$\Pi U \sqsupseteq \Pi V$ implies $U \subseteq V$ for closed $U, V \subseteq W$.*

Proof. (a) From A–1 and A–2, $\mu^*(U) \supseteq \mu^*(V) \Rightarrow \Pi U \sqsupseteq \Pi V$. Since $\mu^*(U)$ and $\mu^*(V)$ are closed (Lemma 3(b)) and because of the Independence Axiom, we obtain $\Pi U \sqsupseteq \Pi V \Rightarrow \Sigma\mu^*(U) \sqsupseteq \Sigma\mu^*(V) \Rightarrow \mu^*(U) \supseteq \mu^*(V)$.
(b) By applying ω^* to both sides of $\mu^*(U) \supseteq \mu^*(V)$, $U \subseteq V$ is proved. □

Lemma 10. *Under a given universe (W, M; A), consider an independent system. For any (U, K) in $2^W \times 2^M$ the following statements are equivalent.*

(1) $K = \mu^*(U)$ *and* $U = \omega^*(K)$.
(2) K *is a closed set and* $U = \omega^*(K)$.
(3) U *is a closed set and* $K = \mu^*(U)$.
(4) U *and* K *are closed sets such that* $\Pi U = \Sigma K$.

Proof. It is easy to see that (1), (2), and (3) are equivalent each other. $(1) \Rightarrow (4)$: By Lemma 3 and A–1. $(4) \Rightarrow (1)$: As U is closed, we have $\omega^*\mu^*(U) = U$. Let $K' = \mu^*(U)$ then $\Sigma K' = \Sigma\mu^*(U) = \Pi U = \Sigma K$. Thanks to the independence assumption, we have $K' = K$. □

4 Semantic similarity lattice alias generalized thesaurus

The consideration in the previous sections leads us to the following reasoning. Under a given universe (W, M; A), the semantics defined by Π and Σ for the elements of 2^W and 2^M through the Axioms A–1, A–2 and A–3 can be suitably expressed as those for closed set pairs (U, K) such that $\Pi U = \Sigma K$ or equivalently $K = \mu^*(U)$ and $U = \omega^*(K)$. When $U \neq \emptyset$ and $K \neq \emptyset$, U is the set of words whose semantics is represented by a set of elementary meanings K. Hence we call this closed set pair (U, K) as a *generalized synonym*.

Define the set $S = \{(U, K) \in 2^W \times 2^M | K = \mu^*(U) \text{ and } U = \omega^*(K)\}$. We can associate an order $\leq$ on S as follows.

For (U_i, K_i) in $S (i = 1, 2)$,

$(U_1, K_1) \leq (U_2, K_2)$ if $K_1 \subseteq K_2$ (or equivalently, $U_1 \supseteq U_2$)

As shown below, (S, $\leq$) becomes a complete lattice which we call as *semantic similarity lattice*. If (W, $\emptyset$) and/or ($\emptyset$, M) are in S, which is the case most of the time, we call $(S - \{(W, \emptyset), (\emptyset, M)\}, \leq)$ as *generalized thesaurus*.

Theorem 11. *($S, \leq$) is a complete lattice in which join and meet are given by*

$$\bigvee_{j \in J}(U_j, K_j) = \left(\bigcap_{j \in J} U_j,\ \mu^*\omega^*\left(\bigcup_{j \in J} K_j\right)\right)$$

$$\bigwedge_{j\in J}(U_j, K_j) = \left(\omega^*\mu^*\left(\bigcup_{j\in J} U_j\right), \bigcap_{j\in J} K_j\right)$$

Proof. It is well known that if U_j's ($j \in J$) are closed, so is $\bigcap_{j\in J} U_j$ [1]. From Lemma 2 we can show that

$$\mu^*\left(\bigcap_{j\in J} U_j\right) = \mu^*\left(\bigcap_{j\in J}(\omega^*\mu^*(U_j))\right) = \mu^*\left(\omega^*\left(\bigcup_{j\in J}(\mu^*(U_j))\right)\right)$$
$$= \mu^*\left(\omega^*\left(\bigcup_{j\in J}(K_j)\right)\right).$$

Therefore $\left(\bigcap_{j\in J} U_j,\ \mu^*\omega^*\left(\bigcup_{j\in J} K_j\right)\right) \in S$. It is trivial that $\left(\bigcap_{j\in J} U_j,\right.$ $\left.\mu^*\omega^*\left(\bigcup_{j\in J} K_j\right)\right)$ is the least upper bound of $\{(U_j, K_j)|j \in J\}$ in S. That is $\vee_{j\in J}(U_j, K_j) = \left(\bigcap_{j\in J} U_j,\ \mu^*\omega^*\left(\bigcup_{j\in J} K_j\right)\right)$. The remaining dual part can be shown similarly. □

Incidentally, this semantic similarity lattice is the dual of a concept lattice [3, 11], as far as the mathematical structure is concerned.

Now we give a simple toy example of universe, generalized synonyms and corresponding semantic similarity lattice for illustrative purpose. The elementary meanings for the chosen words are taken from Webster's Collegiate Thesaurus [5].

Example 1. Consider a universe $(W_1, M_1; A_1)$ where W_1, M_1 and A_1 are given below.

$W_1 = \{$education, information, knowledge, learning, science$\}$
$= \{\mathsf{E}, \mathsf{I}, \mathsf{K}, \mathsf{L}, \mathsf{S}\}$

$M_1 = \{$①, ②, ③, ④, ⑤, , ⑦, ⑧, ⑨$\}$

① : a power or skill that results from persistent endeavor and cultivation
② : enlightenment and excellence of taste acquired by intellectual and aesthetic training
③ : the act or process of educating
④ : the product or result of being educated
⑤ : the quality or state of being erudite
: the act of declaring, proclaiming, or publicly announcing
⑦ : the body of things known about or in science
⑧ : a piece of advice or confidential information given by one thought to have access to special or inside sources
⑨ : a report of events or conditions not previously known

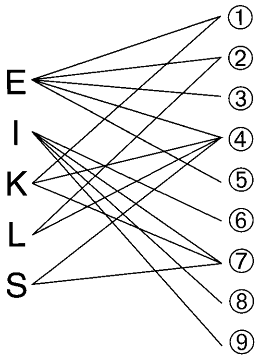

Fig. 1.

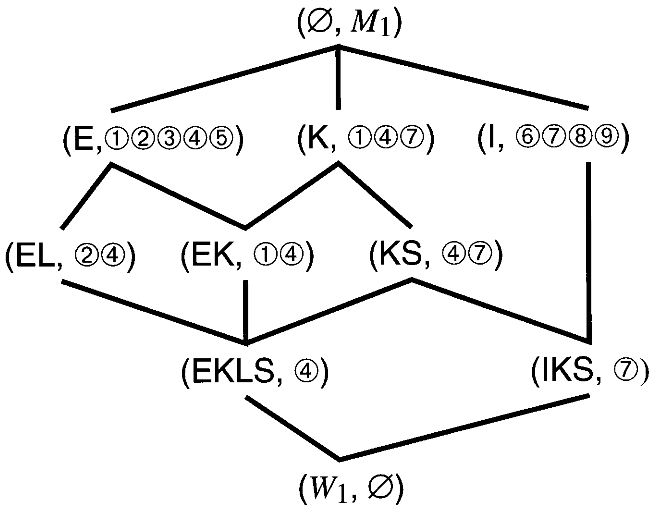

Fig. 2.

A_1:

μ(education) = {①, ②, ③, ④, ⑤}
μ(information) = { , ⑦, ⑧, ⑨}
μ(knowledge) = {①, ④, ⑦}
μ(learning) = {②, ④}
μ(science) = {④, ⑦}

This word-meaning relation A_1 is illustrated as a bipartite graph in Fig. 1. We use the abbreviation like (EL, ②④) in place of ({education, learning}, {②, ④}). The semantic similarity lattice of this universe is illustrated in Fig. 2.

As shown in the lattice diagram, there are 8 generalized synonyms: (E, ①②③④⑤), (K, ①④⑦) (I, ⑦⑧⑨), (EL, ②④), (EK, ①④), (KS, ④⑦), (EKLS, ④), (IKS, ⑦).

Now we can use the lattice $(S, \leq)$ instead of $(R, \sqsubseteq)$ for the semantic analysis of $(W, M; A)$. In order to see this, define a mapping $\Gamma\colon S \to R$ where $\Gamma((U, K)) = \Pi U = \Sigma K$.

Theorem 12. *The mapping Γ is an order preserving mapping from $(S, \leq)$ to $(R, \sqsubseteq)$.*

Proof. From Lemma 10, Γ is well defined. Let $(U_1, K_1) \leq (U_2, K_2)$. Then $K_1 \subseteq K_2$, which implies $\Sigma K_1 \sqsubseteq \Sigma K_2$ by Axiom A–2. That is, $\Gamma((U_1, K_1)) \sqsubseteq \Gamma((U_2, K_2))$. □

Further, define the following two mappings. $\widetilde{\Pi}\colon 2^W \to S$ and $\widetilde{\Sigma}\colon 2^M \to S$ where $\widetilde{\Pi}(U) = (\omega^*\mu^*(U), \mu^*(U))$ for $U \in 2^W$ and $\tilde{\Sigma}(K) = (\omega^*(K), \mu^*\omega^*(K))$ for $K \in 2^M$.

Then we have

Lemma 13.

(a) $\widetilde{\Pi}(U) = \widetilde{\Sigma}\mu^*(U)$ *for any* $U \subseteq W$.
(b) $\widetilde{\Sigma}(K) = \widetilde{\Pi}\omega^*(K)$ *for any* $K \subseteq M$.

Proof. (a) $\widetilde{\Sigma}\mu^*(U) = (\omega^*(\mu^*(U)), \mu^*\omega^*(\mu^*(U))) = (\omega^*\mu^*(U), \mu^*(U)) = \widetilde{\Pi}(U)$.
(b) $\widetilde{\Pi}\omega^*(K) = (\omega^*\mu^*(\omega^*(K)), \mu^*(\omega^*(K))) = (\omega^*(K), \mu^*\omega^*(K)) = \widetilde{\Sigma}(K)$. □

Lemma 14. $K \subseteq L$ *implies* $\widetilde{\Sigma}(K) \leq \widetilde{\Sigma}(L)$ *for any* $K, L \subseteq M$.

Proof. $K \subseteq L \Rightarrow \mu^*\omega^*(K) \subseteq \mu^*\omega^*(L) \Rightarrow (\omega^*(K), \mu^*\omega^*(K)) \leq (\omega^*(L), \mu^*\omega^*(L))$. □

Lemma 15. $\widetilde{\Sigma}(K) \leq \widetilde{\Sigma}(L)$ *implies* $K \subseteq L$ *for any closed* $K, L \subseteq M$.

Proof. Since K and L are closed, we have $\widetilde{\Sigma}(K) \leq \widetilde{\Sigma}(L) \Rightarrow (\omega^*(K), \mu^*\omega^*(K)) \leq (\omega^*(L), \mu^*\omega^*(L)) \Rightarrow (\omega^*(K), K) \leq (\omega^*(L), L) \Rightarrow K \subseteq L$. □

Theorem 16.

(a) $\Pi(U) = \Gamma\widetilde{\Pi}(U)$ *for any* $U \subseteq W$.
(b) $\Sigma(K) = \Gamma\widetilde{\Sigma}(K)$ *for any closed* $K \subseteq M$.

Proof. (a) $\Pi U = \Sigma\mu^*(U) = \Gamma((\omega^*\mu^*(U), \mu^*(U))) = \Gamma\widetilde{\Pi}(U)$
(b) $\Sigma K = \Pi\omega^*(K) = \Gamma((\omega^*(K), \mu^*\omega^*(K))) = \Gamma\widetilde{\Sigma}(K)$. □

These theorems and lemmas show that instead of Π and Σ, we can use $\widetilde{\Pi}$ and $\widetilde{\Sigma}$ respectively, and the set of generalized synonyms can take the place of abstract semantics world. From the viewpoint of the duality of our semantic similarity lattice and the concept lattice of Ganter and Wille, the corresponding part of above discussions about Γ, Π and Σ can be seen in their "The Basic Theorem on Concept Lattice" [3, p.20]. They discuss the conditions that a concept lattice is isomorphic to a complete lattice. In our semantic similarity lattice, we consider a mapping Γ as a relation between a semantic similarity lattice and a partial order set R.

5 Updating universe and generalized thesaurus

In real semantic world, the entities and relations are under constant change: New words are included in the vocabulary, some meanings of words are modified to produce new elementary meanings, and some new connections between words and meanings are established. Then we consider a few primitive operations on the universe to update the existing generalized thesaurus. We here define three operations which update the universe $(W, M; A)$ to $(\overline{W}, \overline{M}; \overline{A})$ with corresponding semantic similarity lattice (SSL) changes. We denote the elementary meaning function and the word function of $(\overline{W}, \overline{M}; \overline{A})$ as $\overline{\mu}$ and $\overline{\omega}$, respectively.

AR) *Add Relation Operation:*
A new word-meaning relation (w, m) is added to A where $w \in W$, $m \in M$ and $(w, m) \notin A$. That is, we consider the case where $\overline{W} = W, \overline{M} = M$ and $\overline{A} = A \cup \{(w, m)\}$.

AW) *Add Word Operation:*
A new word w is added to W where w has a set of elementary meanings N. We assume that N is a subset of M. That is, $\overline{W} = W \cup \{w\} (w \notin W), \overline{M} = M, \overline{A} = A \cup (w \times N), \overline{\mu}(w) = N$.

AM) *Add Meaning Operation:*
A new meaning m is added to M where a set of words V is assumed to have this meaning. That is, $\overline{W} = W$, $\overline{M} = M \cup \{m\} (m \notin M)$, $\overline{A} = A \cup (V \times m)$ and $\overline{\omega}(m) = V$.

We are interested in how the structure of SSL is modified by these operations. More general operations like merging a universe $(W', M'; A')$ with another $(W, M; A)$ can be expressed as combinations of these primitive operations.

Let S $[\overline{S}]$ be the semantic similarity lattice of the universe $(W, M; A)$ $[(\overline{W}, \overline{M}; \overline{A})$, respectively]. For each operation, we give an algorithm to make

$\overline{S}$ from S. In the Appendix, we prove that these algorithms generate $\overline{S}$. In what follows, $\subset$ denotes proper set inclusion.

We define a mapping $\sigma \colon S \to \overline{S}$ to consider the AR operation. We show in the Appendix that σ is a well-defined and order preserving map.

Definition. For $w - m$ connecting AR operation, we define $\sigma \colon S \to \overline{S}$ such that

$$\sigma((U, K)) = \begin{cases} (U, K \cup m) & \text{if} \quad w \in U,\ m \notin K,\ m \in \mu^*(U - w); \\ (U \cup w, K) & \text{if} \quad w \notin U,\ m \in K,\ w \in \omega^*(K - m); \\ (U, K) & \text{otherwise.} \end{cases}$$

Algorithm AR:

1) For each $(U, K) \in S$, $\sigma((U, K)) \in \overline{S}$.
2) For each pair (U_1, K_1) and (U_2, K_2) in S such that (U_2, K_2) covers $(U_1, K_1), U_2 \subset U_2 \cup w \subset U_1$ and $K_1 \subset K_1 \cup m \subset K_2$, let $(U_2 \cup w, K_1 \cup m) \in \overline{S}$.

We say (U_2, K_2) covers (U_1, K_1) if $(U_1, K_1) < (U_2, K_2)$ and there is no generalized synonym of S in between [1]. In step 2) of this algorithm, we can 'fill a gap' between two generalized synonym relations (U_1, K_1) and (U_2, K_2) in S by $(U_2 \cup w, K_1 \cup m)$. In other words, we refine order relation $(U_1, K_1) \leq (U_2, K_2)$ to $(U_1, K_1) \leq (U_2 \cup w, K_1 \cup m) \leq (U_2, K_2)$ if applicable. In the Appendix we show that $\sigma((U_1, K_1)) = (U_1, K_1)$ and $\sigma((U_2, K_2)) = (U_2, K_2)$ in this case.

Algorithm AW:

1) For each $(U, K) \in S$, $(\overline{\omega}^*(K), K) \in \overline{S}$.
2) $(\overline{\omega}^*(N), N) \in \overline{S}$.
3) For each $(U, K) \in S$, $(\overline{\omega}^*(K \cap N), K \cap N) \in \overline{S}$ if $K \cap N \neq \emptyset$.

Step 1) means that if $K \cap N = \emptyset$ then $(U, K) \in \overline{S}$ else $(\overline{\omega}^*(K), K) \in \overline{S}$.

Algorithm AM:

1) For each $(U, K) \in S$, $(U, \overline{\mu}^*(U)) \in \overline{S}$.
2) $(V, \overline{\mu}^*(V)) \in \overline{S}$.
3) For each $(U, K) \in S$, $(U \cap V, \overline{\mu}^*(U \cap V)) \in \overline{S}$ if $U \cap V \neq \emptyset$.

Step 1) means that if $U \cap V = \emptyset$ then $(U, K) \in \overline{S}$ else $(U, \overline{\mu}^*(U)) \in \overline{S}$.

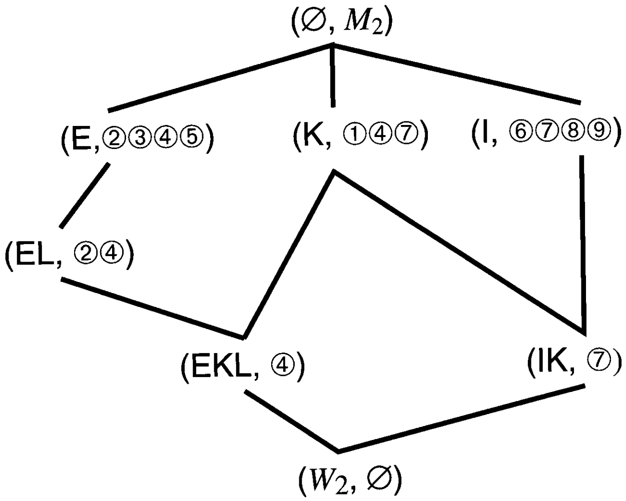

Fig. 3.

Example 2. To illustrate the above algorithms, we start with a somewhat artificial universe (W_2, M_2; A_2) where $W_2 = W_1 - \{\mathsf{S}\}$, $M_2 = M_1$ and $A_2 = A_1 - \{(\{\mathsf{S}\} \times \{④, ⑦\}), (\mathsf{E}, ①)\}$. Semantic similarity lattice of (W_2, M_2; A_2) is shown in Fig. 3.

Example 3 (AW operation). We add a word *science* (S) to the universe $(W_2 M_2; A_2)$ of the above Example 2. That is, $(W_2, M_2; A_2)$ is updated to a new universe $(W_3, M_3; A_3)$ where $W_3 = W_2 \cup \{\mathsf{S}\}$, $M_3 = M_2$, $A_3 = A_2 \cup (\{\mathsf{S}\} \times \{④, ⑦\})$.

In the Algorithm AW step 1, (E, ②③④⑤), (I, ⑦⑧⑨), (K, ①④⑦) and (EL, ②④) are included in $\overline{S}$ without modification. ($\emptyset, M_3$) and ($W_3, \emptyset$) take the place of ($\emptyset, M_2$) and ($W_2, \emptyset$). We add (EKLS, ④) and (IKS, ⑦) corresponding to (EKL, ④) and (IK, ⑦), respectively. (KS, ④⑦) is added in step 2. There is no need to apply step 3 in this case. The whole diagram of $\overline{S}$ is illustrated in Fig. 4.

Example 4 (AR operation). Now we restore the relation (E, ①) to have the universe $(W_1 M_1; A_1)$. That is, $W_1 = W_3$, $M_1 = M_3$, $A_1 = A_3 \cup \{(\mathsf{E}, ①)\}$.

At the step 1 of Algorithm AR, we have $\sigma((\mathsf{E}, ②③④⑤))$=$(\mathsf{E}, ①②③④⑤)$. The other generalized synonyms are included in $\overline{S}$ without modification. A new generalized synonym (EK, ①④) is placed in between the order relation (EKLS, ④) $\leq$ (K, ①④⑦). It is easy to check that this pair satisfies step 2 condition of Algorithm AR. The whole diagram of $\overline{S}$ is in Fig. 2.

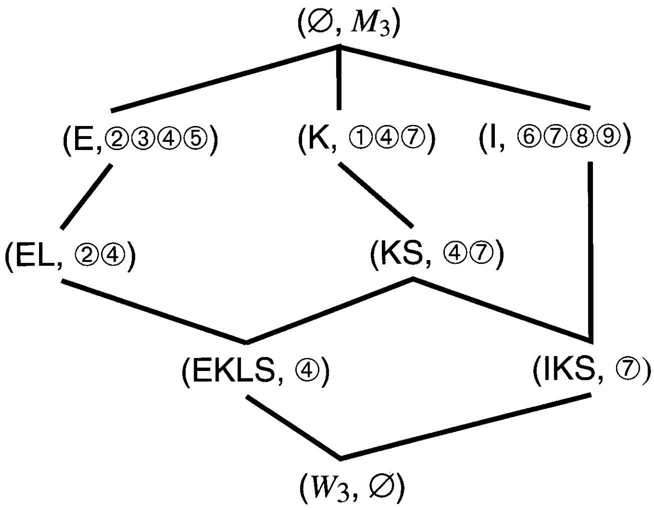

Fig. 4.

6 Concluding remarks

When we build up a database for creative works, an objective flexible and quantitative thesaurus is a must. Many of the existing thesauri for general terms are, however, of more or less subjective nature and categories are selected at compiler's discretion. Also, words and their meanings are under constant changes and once compiled thesaurus has to be kept up with these modifications. Thus more systematic, versatile, and at the same time simple method of thesaurus making seems to be needed.

The concept of elementary meanings has been advocated and utilized in compiling Webster's Collegiate Thesaurus [5]. If each word is supplied with such elementary meanings so that all its meanings are covered by them in a standard fashion, these relations define a Galois pair and naturally yield various closed sets. In this paper, we took an axiomatic way to analyze semantic structure of word groups. Assuming an abstract semantic world, we deduced closed sets as generalized synonym sets. That is, we showed that under certain axioms, we only need to consider closed sets as far as the semantics are concerned. We also showed that the set of generalized synonyms described as a certain pair of closed sets with top and bottom elements makes a complete lattice. Mathematically speaking, our semantic similarity lattice is similar to the concept lattice by Ganter and Wille [3], but our interpretations are rather different.

In order to have a flexible thesaurus, we analyzed structure changes corresponding to three basic environmental changes: A new word-meaning relation is added, a new word or a new meaning is included with its word-meaning relations. Actually we gave algorithms to have updated lattice structure from previous one for the three operations. Although we treated only three additive changes, we can easily introduce similar deletion operations in order to incorporate some missing of the relations.

In this line of investigation, it is of utmost importance to have an appropriate set of elementary meanings for a given set of words and define their relations. This still is a problem to be solved, and to help accomplish this intrinsically hard task, we proposed some axiomatic relations and basic conditions that should hold among the subsets of word set and elementary meaning set, and their correspondence.

We analyzed the relations defined by words-elementary meanings framework. Since this is the first step towards understanding such mathematical structure, we restricted ourselves to the case where only broader/narrower relation is taken care of. The other relations such as antonyms, related terms, and so forth, are left to be incorporated in the future endeavor.

Appendix

We here show somewhat detailed proofs of the validity of update algorithms and related properties described in Sect. 5. We refer to the text for relevant definitions and notations.

Lemma A1. *σ is a well-defined and order preserving mapping from S to $\overline{S}$.*

Proof. First we show that σ is well-defined. That is, we show $\sigma((U,K)) \in \overline{S}$ if $(U,K) \in S$. Let $(U,K) \in S$, and let $w \in U$ and $m \notin K$. Consider $\overline{\mu}$ where $\overline{\mu}$ is the meaning function of universe $(\overline{W}, \overline{M}; \overline{A})$. Since $\overline{\mu} = \mu$ on $W - w$, $\overline{\mu}^*(U) = \overline{\mu}^*(U-w) \cap \overline{\mu}^*(w) = \mu^*(U-w) \cap \{\mu^*(w) \cup m\} = \{\mu^*(U-w) \cap \mu^*(w)\} \cup \{\mu^*(U-w) \cap m\} = K \cup \{\mu^*(U-w) \cap m\}$. Therefore $\overline{\mu}^*(U) = K$ if $m \notin \mu^*(U-w)$ and $\overline{\mu}^*(U) = K \cup m$ if $m \in \mu^*(U-w)$. Because $m \notin K$, $\overline{\omega}^*(K) = \omega^*(K) = U$, we can derive $\overline{\omega}^*(K \cup m) = \overline{\omega}^*(K) \cap \overline{\omega}^*(m) = \omega^*(K) \cap \{\omega^*(m) \cup w\} = \{\omega^*(K) \cap \omega^*(m)\} \cup \{\omega^*(K) \cap w\} = \{U \cap \omega^*(m)\} \cup w$. In the case of $m \in \mu^*(U-w)$, since $\omega^*(m) \supseteq \omega^*\mu^*(U-w) \supseteq U-w$, we have $\{U \cap \omega^*(m)\} \cup w \supseteq \{U \cap \{U-w\}\} \cup w = U$. On the other hand, $\overline{\omega}^*(K \cup m) \subseteq \overline{\omega}^*(K) = \omega^*(K) = U$. Thus $\overline{\omega}^*(K \cup m) = U$. We conclude that $\sigma((U,K)) = (U,K) \in \overline{S}$ if $m \notin \mu^*(U-w)$ and $\sigma((U,K)) = (U, K \cup m) \in \overline{S}$ if $m \in \mu^*(U-w)$.

In the case where $w \notin U$ and $m \in K$, we can prove that $\sigma((U,K)) = (U,K) \in \overline{S}$ if $w \notin \omega^*(K-m)$ and $\sigma((U,K)) = (U \cup w, K) \in \overline{S}$ if $w \in \omega^*(K-m)$, similarly as in the above case.

For the other cases (i.e. $w \notin U$ and $m \notin K$), $\sigma((U,K)) = (U,K) \in \overline{S}$ trivially holds if $(U,K) \in S$.

To prove that σ is order preserving, consider a generalized synonyms pair $(U_1, K_1) \leq (U_2, K_2)$ in S. Only the case where $\sigma((U_1, K_1)) = (U_1, K_1 \cup m)$ and $\sigma((U_2, K_2)) = (U_2 \cup w, K_2)$ is needed to be checked. The other cases are directly proved from the relation $U_1 \supseteq U_2$ or $K_1 \subseteq K_2$. In this case, we have $U_1 \supseteq U_2 \cup w$ from $w \in U_1$, $w \notin U_2$ and $U_1 \supseteq U_2$, and that is $\sigma((U_1, K_1)) \leq \sigma((U_2, K_2))$. □

Lemma A2. *Let* $(U_1, K_1), (U_2, K_2) \in S$ *and* (U_2, K_2) *covers* (U_1, K_1). *That is* $(U_1, K_1) < (U_2, K_2)$ *and there is no generalized synonym of* S *in between. Then for any subset* U_3 *of* W *such that* $U_1 \supset U_3 \supset U_2, \mu^*(U_3) = K_1$ *holds. And also* $\omega^*(K_3) = U_2$ *holds for* $K_3 \subseteq M$ *such that* $K_1 \subset K_3 \subset K_2$.

Proof. From $U_1 \supset U_3 \supset U_2$, we have $K_1 \subseteq \mu^*(U_3) \subseteq K_2$ and $U_1 \supseteq \omega^*\mu^*(U_3) \supseteq U_3 \supset U_2$. Then a generalized synonym $(\omega^*\mu^*(U_3), \mu^*(U_3))$ is in between (U_1, K_1) and (U_2, K_2). Because of the covering assumption, $(\omega^*\mu^*(U_3), \mu^*(U_3)) = (U_1, K_1)$ must hold. That is $\mu^*(U_3) = K_1$. The remaining dual part can be shown similarly. □

Theorem A3. [AR] *Let* $\overline{S}$ *be an SSL of an updated universe* $(\overline{W}, \overline{M}; \overline{A})$ *such that* $\overline{W} = W, \overline{M} = M, \overline{A} = A \cup \{(w, m)\} and (w, m) \notin A$. *That is,* $\overline{S} = \{(U, K) | U \subseteq W, K \subseteq M, U = \overline{\omega}^*(K), K = \overline{\mu}^*(U)\}$. *Then* $\overline{S} = S_\sigma \cup S_{GAP}$ *where*

$$S_\sigma = \sigma(S) = \{\sigma((U, K)) | (U, K) \in S\}$$

and

$$\begin{aligned} S_{GAP} = \{&(U_2 \cup w, K_1 \cup m) | (U_1, K_1), (U_2, K_2) \in S, (U_2, K_2) \\ &\textit{covers } (U_1, K_1), \\ &U_2 \subset U_2 \cup w \subset U_1 \textit{ and } K_1 \subset K_1 \cup m \subset K_2\}. \end{aligned}$$

Proof. From Lemma A1, $\overline{S}$ includes S_σ. So, we show that $\overline{S}$ includes S_{GAP}. Let $(U_2 \cup w, K_1 \cup m) \in S_{GAP}$. Using $\mu^*(U_2 \cup w) = K_1$ from Lemma A2 and $K_1 \cup m \subset K_2 = \mu^*(U_2)$, we obtain $\overline{\mu}^*(U_2 \cup w) = \overline{\mu}^*(U_2) \cap \overline{\mu}^*(w) = \mu^*(U_2) \cap \{\mu^*(w) \cup m\} = \{\mu^*(U_2) \cap \mu^*(w)\} \cup \{\mu^*(U_2) \cap m\} = K_1 \cup m$. $\overline{\omega}^*(K_1 \cup m) = U_2 \cup w$ can be proved by the similar way. Therefore $(U_2 \cup w, K_1 \cup m) \in \overline{S}$.

On the other hand, we can show that a generalized synonym of $\overline{S}$ is an element of S_σ or S_{GAP}. Let $(U, K) \in \overline{S}$. We consider the case where $U \neq \emptyset$ and $K \neq \emptyset$, first. If $w \notin U$ holds then $\mu^*(U) = \mu^*(U) = K$. If $m \notin K$, then $\omega^*(K) = U$ holds trivially. Even if $m \in K$ holds, we can show that $\omega^*(K) = \omega^*(\{K - m\} \cup m) = \omega^*(K - m) \cap \omega^*(m) = \overline{\omega}^*(K - m) \cap \{\overline{\omega}^*(m) - w\} = \{\overline{\omega}^*(K - m) \cap \overline{\omega}^*(m)\} - w = \overline{\omega}^*(K) - w = U$. That is $(U, K) \in S$, if $w \notin U$ holds. It can be proved similarly that $(U, K) \in S$ if

$m \notin K$. So we concentrate on the case where $w \in U$ and $m \in K$. Let $U = U_2 \cup w$ and $K = K_1 \cup m$ such that $w \notin U_2$ and $m \notin K_1$. Since w is not related to m in $(W, M; A)$, $(U, K) = (U_2 \cup w, K_1 \cup m) \notin S$. Then $\mu^*(U_2 \cup w) = \mu^*(U_2) \cap \mu^*(w) = \overline{\mu}^*(U_2) \cap \{\overline{\mu}^*(w) - m\} = \{\overline{\mu}^*(U_2) \cap \overline{\mu}^*(w)\} - m = K_1$. $\omega^*(K_1 \cup m) = U_2$ can be proved by the similar way. Let $K_2 = \mu^*(U_2)$ and $U_1 = \omega^*(K_1)$. Since U_2 and K_1 are closed sets of $(W, M; A)$, we have $(U_2, K_2) \in S$ and $(U_1, K_1) \in S$. Applying the closure operator $\omega^*\mu^*$ to $U_2 \cup w$, we obtain $U_2 \cup w \subseteq \omega^*\mu^*(U_2 \cup w) = \omega^*(K_1) = U_1$. Therefore $U_2 \subset U_2 \cup w \subseteq U_1$. In the same manner, $K_1 \subset K_1 \cup m \subseteq K_2$ is proved. If $U_2 \cup w = U_1$ holds, the condition $m \in K_2 = \mu^*(U_2) = \mu^*(U_1 - w)$ is satisfied. Then we have $(U, K) = (U_2 \cup w, K_1 \cup m) \in \overline{S}$ and $(U_2 \cup w, K_1) = (U_1, K_1) \in S$, which corresponds to the case where (U, K) is an element of S_σ. And $(U, K) \in S_\sigma$ is proved similarly if $K_1 \cup m = K_2$. The case where $U_2 \cup w \subset U_1$ and $K_1 \cup m \subset K_2$ is nothing but the case where $(U, K) \in S_{GAP}$. Note that $\sigma((U_1, K_1)) = (U_1, K_1)$ and $\sigma((U_2, K_2)) = (U_2, K_2)$ are proved through the above discussion. For the other cases, including the case when $U = \emptyset$ or $K = \emptyset$, it is trivial that $(U, K) \in S_\sigma$. □

Theorem A4. [AW] *Let $\overline{S}$ be an SSL of an updated universe $(\overline{W}, \overline{M}; \overline{A})$ such that $\overline{W} = W \cup \{w\}$, $\overline{M} = M$, $\overline{A} = A \cup \{w \times N\}$, $\overline{\mu}(w) = N$. That is, $\overline{S} = \{(U, K) | U \subseteq \overline{W}, K \subseteq \overline{M}, U = \overline{\omega}^*(K), K = \overline{\mu}^*(U)\}$. Then*

$$\overline{S} = \{(\overline{\omega}^*(K), K) | (\omega^*(K), K) \in S\} \cup \{(\overline{\omega}^*(N), N)\}$$
$$\cup \{(\overline{\omega}^*(K_1 \cap N), K_1 \cap N) | (\omega^*(K_1), K_1) \in S, K_1 \cap N \neq \emptyset\}.$$

Proof. First, we show that $\overline{S}$ has these generalized synonyms. Let $(U, K) \in S$ where $U \neq \emptyset$ and $K \neq \emptyset$. Since $w \notin U$, $\overline{\mu}^*(\overline{\omega}^*(K)) = \overline{\mu}^*(\overline{\omega}^*(\mu^*(U))) = \overline{\mu}^*(\overline{\omega}^*(\overline{\mu}^*(U))) = \overline{\mu}^*(U) = \mu^*(U) = K$. Therefore $(\overline{\omega}^*\overline{\mu}^*(U), K) = (\overline{\omega}^*(K), K) \in \overline{S}$. Also $(\overline{\omega}^*\overline{\mu}^*(w), \overline{\mu}^*(w)) = (\overline{\omega}^*(N), N) \in \overline{S}$. Because $\overline{S}$ is a complete lattice, $(\overline{\omega}^*(K), K) \wedge (\overline{\omega}^*(N), N) = (\overline{\omega}^*(K \cap N), K \cap N) \in \overline{S}$ for all $(U, K) \in S$ when $K \cap N \neq \emptyset$.

Let $U = \emptyset$ and let $(\emptyset, M) \in S$. If $\overline{\mu}(w) = N = M$ then $\overline{\omega}^*(M) = w$ and $(w, M) \in \overline{S}$. If $\overline{\mu}(w) = N \neq M$ then $(\emptyset, M) \in \overline{S}$.

Let $K = \emptyset$ and assume that $(W, \emptyset) \in S$. $(W \cup \{w\}, \emptyset) = (\overline{W}, \emptyset) \in \overline{S}$ is trivial.

Next, we prove that a synonym relation in $\overline{S}$ has one of these expressions. Let $(U, K) \in \overline{S}$, $U \neq \emptyset$ and $K \neq \emptyset$. If $w \notin U$ then $K = \overline{\mu}^*(U) = \mu^*(U)$. That is K is closed in $(W, M; A)$. Then $(\omega^*(K), K) \in S$. In the case where $w \in U, K = \overline{\mu}^*(U) = \overline{\mu}^*(U - w) \cap \overline{\mu}^*(w) = \mu^*(U - w) \cap N$. Let $K_1 = \mu^*(U - w)$. Because K_1 is closed in $(W, M; A)$, $(\omega^*(K_1), K_1) \in S$. Since $U = \overline{\omega}^*(K) = \overline{\omega}^*(K_1 \cap N)$, $(U, K) = (\overline{\omega}^*(K_1 \cap N), K_1 \cap N) = (\overline{\omega}^*(K_1), K_1) \wedge (\overline{\omega}^*(N), N)$. In the case where $K = K_1 \cap N = N$, it is clear that $(U, K) = (\overline{\omega}^*(N), N)$.

$(\omega^*(M), M) \in S$ corresponds to $(\overline{\omega}^*(M), M) \in \overline{S}$ and $(W, \mu^*(W)) \in S$ to $(\overline{W}, \overline{\mu}^*(\overline{W})) \in \overline{S}$. □

Theorem A5. [AM] *Let $\overline{S}$ be an SSL of an updated universe $(\overline{W}, \overline{M}; \overline{A})$ such that $\overline{W} = W$, $\overline{M} = M \cup \{m\}$, $\overline{A} = A \cup \{V \times m\}$, $\overline{\omega}(m) = V$. That is, $\overline{S} = \{(U, K) | U \subseteq W, K \subseteq \overline{M}, U = \overline{\omega}^*(K), K = \overline{\mu}^*(U)\}$. Then*

$$\overline{S} = \{(U, \overline{\mu}^*(U)) | (U, \mu^*(U)) \in S\} \cup \{(V, \overline{\mu}^*(V))\}$$
$$\cup \{(U_1 \cap V, \overline{\mu}^*(U_1 \cap V)) | (U_1, \mu^*(U_1)) \in S, U_1 \cap V \neq \emptyset\}.$$

Proof. From the duality of μ and ω, this theorem is proved like Theorem A4. □

Acknowledgements. The authors are grateful to Profs. T. Sakai (Ryukoku University) and O. Kakusho (Hyogo University) for their support and encouragement during various stages of this work.

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